imaginary pair of complex eigenvalues at $\pm j\omega$, providing $c_2 a_1 > 0$. It remains to enforce the conditions of property II. To find ξ_0 and η_0 it is necessary to solve

$$A\xi_0 = i\omega\xi_0 \text{ and } A'\eta_0 = i\omega\eta_0$$

Carrying out the calculation, it turns out that

$$\xi_0 = \begin{bmatrix} b_1 \\ b_2 \\ 1 \\ i\omega \end{bmatrix} \qquad \eta_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ i\omega \end{bmatrix}$$

where b_1, b_2 are complex constants depending on a_0, a_1, ω, and c_2. Consequently, the dot product conditions become

$$\langle B\xi_0, \eta_0 \rangle = 2\zeta\omega \neq 0$$

and $\langle N(\xi_0), \eta_0 \rangle \neq 0$ if

$$3a_1^2\omega^2 \neq (a_0 b_1 + a_0 + a_1 b_2)^2$$
$$a_0 b_1 + a_0 + a_1 b_2 \neq 0$$

Conclusion

In the preceding, we have assumed a cubic nonlinearity in the rudder dynamics and have determined conditions under which a bifurcating branch of orbitally stable periodic solutions will exist. In the case considered, it was possible to determine rather easily conditions under which the system matrix had a pair of simple, pure imaginary, eigenvalues. In more complicated cases, this can still be accomplished by utilizing various linear stability techniques. The D-decomposition method of determining stability regions[5] ought to prove especially useful in this application.

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Technical Comments

Comment on "Inclusion of Transverse Shear Deformation in Finite Element Displacement Formulations"

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IN a recent Note,[1] the stiffness matrix of a beam element including the shear deformation has been derived by a displacement formulation. The interesting point in that derivation is the necessity to distinguish between the first derivative of the transverse displacement and the rotation of the normal to the cross section of the beam. It is interesting to show that the same result can be easily obtained by using the flexibility matrix of a beam finite element.

The relations between coordinate displacements and forces in Fig. 1 are given by

$$\delta = \alpha S \tag{1}$$

where the flexibility matrix α is obtained from the following expression for the stress energy

$$U' = \frac{1}{2EI}\int_0^a M^2 dx + \frac{1}{2kGA}\int_0^a V^2 dx \tag{2}$$

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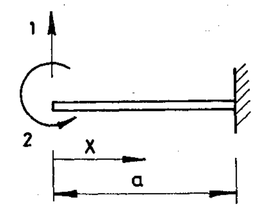

Fig. 1 Coordinates S, δ, for a beam finite element.

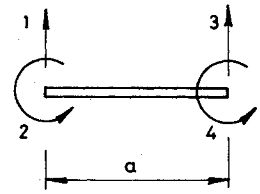

Fig. 2 Coordinates F, u, for a free beam finite element.

From Eq. (2) one obtains

$$[\alpha] = \frac{a}{EI}\begin{bmatrix} a^2/3 + g & -a/2 \\ -a/2 & 1 \end{bmatrix} \tag{3}$$

where, as in Ref. 1, $g = EI/kGA$ and k is the shear factor. From equilibrium considerations, the relationship between the forces in Figs. 1 and 2 is given by

$$F = \beta S \tag{4}$$

where

$$[\beta] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ a & -1 \end{bmatrix} \tag{5}$$

By equating the strain energy in Figs. 1 and 2 the stiffness matrix defined by the relation, $F \doteq ku$, is obtained

$$[k] = \beta\, \alpha^{-1} \beta^T \tag{6}$$

$$[k] = \frac{12}{a^2 + 12g}\,\frac{EI}{a} \begin{bmatrix} 1 & & & \text{sym} \\ a/2 & a^2/3+g & & \\ -1 & -a/2 & 1 & \\ a/2 & a^2/6-g & -a/2 & a^2/3+g \end{bmatrix} \tag{7}$$

The stiffness matrix given in Eq. (7) is exactly the same as the one obtained in Ref. 1 using a displacement formulation.

The important conclusion from the derivation given in this comment is that by using the flexibility matrix one can include the shear deformation without worrying about the differences between the first derivative of the transverse displacement and the rotation. These differences are distinguished automatically.

References

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Reply by Authors to M. Baruch

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M. BARUCH has correctly pointed out that inclusion of transverse shear deformation in finite element force (flexibility) formulations does not require choosing between the first derivatives of the transverse displacements and the rotations of the normals to the neutral surface of the finite element as degrees of freedom. Once the flexibility matrix has been derived in terms of an appropriate set of forces and moments and inverted to form the stiffness matrix, the generalized coordinates associated with the rows and columns of the stiffness matrix are guaranteed to be the correct variables. Incidently, the flexibility matrix derived by Baruch may also be found in Ref. 1.

The point of Ref. 2 as stated in the conclusions was that, in contrast to previous assertions,[3,4] the finite element *displacement* approach proceeds by a straight-forward energy minimization to yield the correct element stiffness matrix even when transverse shear deformation is included. To obtain the correct result by the displacement approach, however, one must use the correct rotational degrees of freedom, namely the rotations of the normals to the middle surface.

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Comment on "Stability of a Spinning Satellite with Flexible Antennas"

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A RECENT paper by Dong and Schlack[1] presents a stability analysis of a spinning flexible satellite via the assumed modes method, whereby the displacements of the flexible parts are represented by series of space-dependent admissible functions multiplied by time-dependent generalized coordinates. As admissible functions they use the fixed-base cantilever modes. Neither the procedure nor the results are new, however.

The problem considered in Ref. 1 was first investigated by Meirovitch and Nelson,[2] who used the assumed modes method in conjunction with an infinitesimal analysis to test the stability of precisely the same mathematical model as that of Ref. 1. In fact, the parameter plot of Ref. 1 (Fig. 2) was originally presented in Ref. 2 (Fig. 2). The stability analysis of both a torque-free spinning flexible satellite and a gravity-gradient stabilized flexible satellite was performed by Meirovitch and Calico[3,4] via the Liapunov direct method in conjunction with the assumed modes method (in addition to the method of integral coordinates and the method of testing density functions). Although Ref. 3 is concerned with a more complicated mathematical model than that of Ref. 2 (and hence than that of Ref. 1), in the sense that the system contains four radial booms in addition to the two axial booms, it does consider also the mathematical model of Ref. 2 for comparison purposes.

The authors of Ref. 1 imply that the ability to derive closed-form stability criteria in terms of infinite series represents a new development. A careful examination of both Refs. 3 and 4, however, reveals that the closed-form stability criteria derived in these references by the assumed modes method are indeed in terms of infinite series. In fact, they correspond exactly to those of Ref. 1, as for the assumed modes method the bounding properties of Rayleigh's quotient need not be used and were in fact not used. The Hessian matrix tested for sign definiteness can be shown[5] to have the form

$$[H]_E = [I + b_i \delta_{ij}]$$

where $b_i (i=0,1,2,...)$ are real numbers depending on the system parameters. Using Sylvester's criterion, and con-

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